JOURNAL OF FUNCTIONAL ANALYSIS 21, 380–388 (1976)

Interpolating Sequences and the Shilov Boundary of $H^\infty(\Delta)$*

STEVEN ZISKIND

Department of Mathematics, University of California at Los Angeles

Communicated by John Wermer

Received November 5, 1974

Let $H^\infty(\Delta)$ denote the Banach algebra of bounded analytic functions on the open unit disc, let $\mathscr{M}$ denote its maximal ideal space, and let ∂ denote its Shilov boundary. D. J. Newman has shown that a homomorphism φ in $\mathscr{M}$ will be in ∂ if and only if φ is unimodular on all Blaschke products. We answer a question of K. Hoffman by showing that φ will be in ∂ if and only if φ is unimodular on every Blaschke product whose zero set is an interpolating sequence. Our method is based on a construction due to L. Carleson, originally developed for the proof of the Corona theorem.

1. INTRODUCTION

Let Δ denote the open unit disc in $\mathbb{C}$ and let $H^\infty(\Delta)$ denote the Banach algebra of bounded analytic functions on Δ. The algebra $H^\infty(\Delta)$ has a maximal ideal space, which we denote by $\mathscr{M}$, and a Shilov boundary, which we denote by ∂. Newman [6, p. 179] has shown that for φ in $\mathscr{M}$, the following are equivalent:

(i) $\varphi \in \partial$

(ii) $|\varphi(B)| = 1$ whenever B is a Blaschke product

(iii) $\varphi(B) \neq 0$ whenever B is a Blaschke product.

Call a sequence $\{z_j\}_{j=1}^\infty$ in Δ an *interpolating sequence* if for every bounded sequence $\{a_j\}_{j=1}^\infty$ there is an f in $H^\infty(\Delta)$ for which $f(z_j) = a_j$ for all j. Every interpolating sequence must satisfy $\Sigma(1 - |z_j|) < \infty$, and so is the zero sequence for a Blaschke product. We call a Blaschke product with interpolating zero sequence an *interpolating Blaschke*

* This paper is based on the author's Ph.D. thesis prepared at U.C.L.A. under the direction of Professor John Garnett.

product. Hoffman [9] has asked the following question: If we know that $|\varphi(B)| = 1$ for every interpolating Blaschke product B, can we conclude that φ is in ∂? We answer this question affirmatively.

THEOREM 1. *Let $\varphi \in \mathscr{M}$ be such that $|\varphi(B)| = 1$ whenever B is an interpolating Blaschke product. Then $\varphi \in \partial$.*

Theorem 1 is actually proved in a somewhat stronger form (stronger in view of Newman's result).

THEOREM 1S. *Let φ be in $\mathscr{M} \backslash \partial$. Then for every $\epsilon > 0$, there is a corresponding interpolating Blaschke product B^* for which $|\varphi(B^*)| \leqslant \epsilon$.*

The proof of Theorem 1S will be based on the following result.

THEOREM 2. *Let B be any Blaschke product on Δ, and let $\epsilon > 0$. Then there is a corresponding interpolating Blaschke product B^* for which $|B^*(z)| < \epsilon$ whenever $|B(z)| < 2^{-M}$ (where M is a universal constant). Moreover, all the zeros of B^* with modulus at least $\frac{1}{2}$ will be in the set $\{z : |B(z)| \leqslant \frac{1}{4}\}$.*

The proof of Theorem 2 uses a construction from the proof of the Corona theorem and will be deferred to Sections 4, 5 and 6.

2. PROOF OF THEOREM 1S

Let us deduce Theorem 1S from Theorem 2.

Proof of Theorem 1S. Let φ be in $\mathscr{M} \backslash \partial$. By Newman's theorem, there is a Blaschke product B for which $\varphi(B) = 0$. Let B^* be the corresponding interpolating Blaschke product from Theorem 2. By the Corona theorem, we may choose a net of points $\{z_\alpha\} \subset \Delta$ whose corresponding point evaluations converge weak-star to φ. Then $B(z_\alpha) \to \varphi(B) = 0$ and $|B^*(z_\alpha)| \to |\varphi(B^*)|$. Since $|B(z_\alpha)| < 2^{-M}$ for α sufficiently large, we also have $|B^*(z_\alpha)| < \epsilon$ for α sufficiently large. Consequently $|\varphi(B^*)| \leqslant \epsilon$.

It is interesting to note that a weakened version of Theorem 2 is equivalent to Theorem 1.

THEOREM 2W. *Let B be any Blaschke product and let $0 < \epsilon < 1$. Then there is some interpolating Blaschke product B_α for which*

$$\sup_{\{z : |B(z)| < \epsilon\}} |B_\alpha(z)| < 1.$$

Proof. Suppose, in contradiction, that for every interpolating Blaschke product B_α, we do have $\sup_{\{z:|B(z)|<\epsilon\}} |B_\alpha(z)| = 1$. Let $\mathscr{B}$ denote the collection of all finite products of interpolating Blaschke products which do not vanish at the origin. Then for every B_α in $\mathscr{B}$, there is some point z_α in Δ for which $|B(z_\alpha)| < \epsilon$ and

$$|B_\alpha(z_\alpha)| \geqslant 1 - |B_\alpha(0)|.$$

Partially order $\mathscr{B}$ by containment of zero sets (counting multiplicity). Explicitly, $\alpha > \beta$ if $B_\alpha = B_\beta \cdot B_\gamma$ for some B_γ in $\mathscr{B}$. $\{z_\alpha\}$ is now a net in Δ. We claim that for each fixed α, $\lim_\gamma |B_\alpha(z_\gamma)| = 1$. To see this, let $\delta > 0$ be given and choose $\gamma_0 > \alpha$ so that $|B_\gamma(0)| < \delta$ whenever $\gamma > \gamma_0$. Then $\gamma > \gamma_0$ implies that

$$|B_\alpha(z_\gamma)| \geqslant |B_\gamma(z_\gamma)| \geqslant 1 - |B_\gamma(0)| > 1 - \delta$$

as desired. The compactness of $\mathscr{M}(H^\infty)$ guarantees that $\{z_\alpha\}$ has a subnet converging to some φ in $\mathscr{M}(H^\infty)$. We now have $|\varphi(B)| \leqslant \epsilon < 1$ and $|\varphi(B_\alpha)| = 1$ for every B_α in $\mathscr{B}$. In particular, φ cannot be evaluation at any point in Δ. Thus $|\varphi(z)| = 1$, where z is the identity function in $H^\infty(\Delta)$. Now any interpolating Blaschke product is the product of a function in $\mathscr{B}$ and the function z (possibly). Thus, $|\varphi(B_\alpha)| = 1$ for all interpolating Blaschke products B_α. This contradicts Theorem 1, proving Theorem 2W.

The proof that Theorem 2W implies Theorem 1 is nearly a verbatim copy of the proof of Theorem 1S.

It is natural to ask if Theorem 1S is sharp. Explicitly, if φ is not in ∂, is there an interpolating Blaschke product B^* for which $\varphi(B^*) = 0$? In general such a B^* will not exist. For suppose there were an interpolating Blaschke product B^* for which $\varphi(B^*) = 0$. Then φ is in the closure of the zero sequence of B^* (see [6, p. 206]). Hoffman [7] has shown, however, that this happens precisely when φ has nontrivial Gleason part. Since there are trivial Gleason parts in $\mathscr{M}\backslash\partial$ [7], there may be no B^* for which $\varphi(B^*) = 0$. This observation is due to Gamelin.

3. Applications to Douglas Algebras

Let us recall that a subalgebra $\mathfrak{A}$ of $L^\infty(\partial\Delta)$ is called a Douglas algebra if $\mathfrak{A}$ is the closed algebra generated by $H^\infty(\partial\Delta)$ and $\{\bar{B} \in \mathfrak{A} : B$ is a Blaschke product$\}$. D. Sarason (private communication) has asked whether the set $\{\bar{B}^* \in \mathfrak{A} : B^*$ is an interpolating Blaschke product$\}$

can replace the corresponding set above. Donald E. Marshall (private communication) has provided the following affirmative answer.

THEOREM 3. *If $\mathfrak{A}$ is any Douglas algebra, then $\mathfrak{A}$ is the closed algebra generated by $H^\infty(\partial\Delta)$ and $\{\bar{B}^* \in \mathfrak{A} : B^*$ is an interpolating Blaschke product$\}$.*

The proof of Theorem 3 is a simple result of the following.

LEMMA. *Let B be any Blaschke product and let B^* be a corresponding interpolating Blaschke product from Theorem 2. Let $[H^\infty, \bar{B}]$ and $[H^\infty, \bar{B}^*]$ denote the Douglas algebras they generate. Then*

$$[H^\infty, \bar{B}] = [H^\infty, \bar{B}^*].$$

Proof. The maximal ideal space of $[H^\infty, \bar{B}]$, denoted $M([H^\infty, \bar{B}])$ is $\{\varphi \in \mathscr{M} : |\varphi(B)| = 1\}$ (see [5]). Similarly,

$$M([H^\infty, \bar{B}^*]) = \{\varphi \in \mathscr{M} : |\varphi(B^*)| = 1\}.$$

Now $\bar{B} \in [H^\infty, \bar{B}^*]$ if and only if B is invertible in $[H^\infty, \bar{B}^*]$. This happens if and only if $\hat{B}$ is never zero on $M([H^\infty, \bar{B}^*])$. This latter condition fails if and only if there is a φ in $\mathscr{M}$ for which $\varphi(B) = 0$ and $|\varphi(B^*)| = 1$. The same argument that proves Theorem 1 shows that this cannot happen. Thus $\bar{B} \in [H^\infty, \bar{B}^*]$.

Similarly $\bar{B}^* \notin [H^\infty, \bar{B}]$ if and only if there is some φ in $\mathscr{M}$ for which $|\varphi(B)| = 1$ and $\varphi(B^*) = 0$. But if $\varphi(B^*) = 0$, then φ is in the closure of the zero set of B^* which is contained in $\{z : |B(z)| < \frac{1}{4}\}$. This would say that $|\varphi(B)| \leqslant \frac{1}{4}$. Consequently $\bar{B}^* \in [H^\infty, \bar{B}]$, and so the lemma is proved.

4. Carleson Measures and the ρ-Metric

Throughout the remainder of this paper, we will work in both the disc Δ, and the upper half-plane, H^+. Ultimately our results will be stated for Δ, but some of the constructions are best explained in H^+.

DEFINITION. A finite measure μ on Δ is called a *Carleson measure* if there is a constant K for which $\mu(S) \leqslant K \cdot l$ whenever S is a sector of the form $S = \{re^{i\theta} : 1 - l \leqslant r < 1;\ \theta_0 \leqslant \theta \leqslant \theta_0 + l\}$. The analogous definition is made for H^+ also, where squares replace sectors.

Any rectifiable curve $\Gamma \subset \Delta$ (resp. H^+) induces a finite measure on Δ (resp. H^+) by defining the measure of any Borel set S to be the length of $\Gamma \cap S$. We say Γ induces a Carleson measure if the induced measure is Carleson.

DEFINITION. The *hyperbolic distance* between two points is defined by

$$\rho(z, w) = \begin{cases} \left| \dfrac{z - w}{1 - \bar{w}z} \right| & \text{on } \Delta, \\ \left| \dfrac{z - w}{z - \bar{w}} \right| & \text{on } H^+. \end{cases}$$

The relevance of the ρ-metric and of Carleson measures to our problem rests in the following characterization of interpolation [2]. On Δ, a sequence $\{z_j\}$ is interpolating if and only if there is some $\epsilon > 0$ for which $\rho(z_j, z_k) \geqslant \epsilon$ for $j \neq k$ and if the measure

$$\Sigma(1 - |z_j|)\, \delta_{z_j}$$

is a Carleson measure. (On H^+, we use instead the measure $\Sigma(\operatorname{Im} z_j)\, \delta_{z_j}$.)

Our plan of attack on Theorem 2 runs as follows. In Section 5, we let B be any Blaschke product and construct a family of contours in Δ surrounding the places where $|B(z)| < 2^{-M}$. Calling Γ the union of the contours, we show that Γ induces a Carleson measure on Δ. This construction is due to Carleson [3]. It was originally stated only for finite Blaschke products but the proof carries over to the infinite case after some modifications. In Section 6, we take the Γ from Section 5 and let $\{a_n\}$ be a set of points on Γ which are uniformly distributed in the ρ-metric. The properties of Γ will guarantee that $\{a_n\}$ is an interpolating sequence, and the Blaschke product B^* with zeros at $\{a_n\}$ will be shown to have the properties required by Theorem 2.

5. THE CONSTRUCTION OF Γ

LEMMA 1. *Let $f\colon \Delta \to \Delta$ be analytic and let $|f(0)| = a > 0$. For each $M > 0$, define*

$$E_M = \{\theta : \inf_{0 \leqslant r < 1} |f(re^{i\theta})| < e^{-M}\}.$$

Then, given any $\epsilon > 0$, there is a constant $M(a, \epsilon)$ such that $|E_M| < \epsilon$ whenever $M > M(a, \epsilon)$, where $|E_M|$ denotes arc length of E_M.

LEMMA 2. *There is a constant M_0 such that whenever*

(1) $f\colon H^+ \to \Delta$ *is a Blaschke product,*

(2) $Q = \{(x, y)\colon x_0 < x < x_0 + \delta, 0 < y < \delta\}$,

(3) *there is a point* $z_0 \in Q$ *with* $\operatorname{Im} z_0 > \delta/2$ *and* $|f(z_0)| > \frac{1}{4}$,

then

$$|\{z \in Q : |f(z)| < 2^{-M_0}, \operatorname{Im} z \leqslant \delta/4\}^*| < \delta/2,$$

where A^* denotes projection of A onto $\{y = 0\}$.

Lemma 1 is proved in [4] and Lemma 2 follows from Lemma 1 by conformal mapping. Now let $Q^0 = \{(x, y)\colon 0 \leqslant x \leqslant 1, 0 < y \leqslant 1\}$. The top half of Q^0 will be called R^0. $Q^0 \backslash R^0$ is partitioned into two squares, each of which we call Q^1. The top half of each Q^1 is called R^1. Continue the process indefinitely. Let $m = \max(M_0, 50)$ and partition each R^k into closed dyadic squares of edge length 2^{-m-k}. Call each such square S. For any fixed S, simple estimates show that

$$\sup_{z,w \in S} \rho(z, w) < 2 \cdot 2^{-m}. \tag{†}$$

An application of the Schwarz–Pick lemma [1] to (†) yields the following.

LEMMA 3. *Let $f\colon H^+ \to \Delta$ be analytic, let $z_0 \in S$, and let $|f(z_0)| < 2^{-m}$. Then*

$$\sup_S |f(z)| < 4 \cdot 2^{-m}.$$

Suppose now that f is a Blaschke product defined on H^+. We wish to choose a subcollection, $\mathscr{C}$, of the squares S, with the property that

(α) if $z \in S$ and $|f(z)| < 2^{-m}$ then $S \in \mathscr{C}$;

(β) if $z \in S$ and $|f(z)| > \frac{1}{4}$ then $S \notin \mathscr{C}$.

Lemma 3 guarantees that (α) and (β) will be mutually exclusive. To decide which of the S go into $\mathscr{C}$, we consider two cases.

Case I. $\sup_R |f| \geqslant \frac{1}{4}$. For $S \subset R^0 \cup R^1$, put S in $\mathscr{C}$ if

$$\inf_S |f| \leqslant 2^{-m}.$$

For $S \subset R^2$, put S in $\mathscr{C}$ if $\inf_S |f| \leqslant 2^{-m}$, and shade the square $Q(S)$ whose base is S^* (unless it has already been shaded). For $S \subset R^k$ with $k > 2$, put S in $\mathscr{C}$ if $\inf_S |f| \leqslant 2^{-m}$ and if $S \not\subset Q(S_0)$ for some $S_0 \in R^j$ with $j < k$. Also shade $Q(S)$ if it does not already lie in a shaded square. Label the shaded squares $Q_1^{(1)}$, $Q_2^{(1)}$, $Q_3^{(1)}$,... and call them *first generation squares*. Because of Lemma 2,

$$\left|\left(\bigcup_{j=1}^{\infty} Q_j^{(1)}\right)^*\right| < \tfrac{1}{2}\,|(Q^0)^*|,$$

and we can easily check that

$$\sum_{Q(S_0)\text{ below } S \text{ below } S_0} p(S) \leqslant \text{const}\,|S_0{}^*| \tag{*}$$

where $p(S)$ is the perimeter of S and the constant depends only on m. The subcollection of the S in $\mathscr{C}$ which are placed in $\mathscr{C}$ by Case I will be called G.

Case II. $\sup_{R^0}|f| < \frac{1}{4}$. First place all of the $S \subset R^0$ into $\mathscr{C}$. Next consider each of the squares Q^1. If $|f| > \frac{1}{4}$ anywhere in the top half of Q^1, shade that Q^1. Otherwise put all the S in the top half of Q^1 into $\mathscr{C}$ and repeat the process on the two Q^2's which form the bottom half of Q^1. Continue this process as long as possible outside the shaded squares. As before, the shaded squares are called *first generation squares* and labelled $Q_1^{(1)}$, $Q_2^{(1)}$,... . Let T be the union of the S outside of these shaded squares. Note that T is simply connected, that

$$|\partial T| < 6\,|(Q^0)^*|, \tag{**}$$

and that the first generation squares produced by this case are squares to which Case I applies.

Given a Blaschke product f defined on H^+, proceed as follows. Start with Case I or II and then pass to the shaded squares $Q_1^{(1)}$, $Q_2^{(1)}$,... . On each $Q_j^{(1)}$, apply the appropriate case and obtain doubly shaded squares $Q_1^{(2)}$, $Q_2^{(2)}$,... which we call *second generation squares.* Repeat the process indefinitely. Let $\mathscr{G}$ denote the collection of all the T's formed by the Case II construction. Because we never use Case II twice consecutively in passing from one shaded square to a shaded descendant, our choice of m guarantees that

$$\sum_{j=0}^{\infty} |(Q_j^{(2n)})^*| \leqslant 2^{-n}.$$

Combined with (*) and (**) we can now easily check that Γ, defined as the union of the boundaries of the S in G and the T in $\mathscr{G}$, induces a Carleson measure.

Note further that $|\partial T \cap \{y = 0\}| = 0$ for every T in $\mathscr{G}$, because Blaschke products have unimodular vertical limits a.e. and any point

$(x, 0) \in \partial T$ is a point where $\overline{\lim}_{y \to 0} |f(x, y)| \leqslant \frac{1}{4}$. Finally, note that any point z in Q^0 for which $|f(z)| < 2^{-m}$ will be surrounded by Γ.

6. The Construction of B^*

In this section we consider a curve $\Gamma \subset Q^0$ which is of the type produced in Section 5. We want to analyze the behavior of a Blaschke product whose zeros are located on Γ. Explicitly, choose any ϵ for which $0 < \epsilon < 2^{-m-1}$ and place points a_n on Γ so that

$$\epsilon \leqslant \rho(a_n, a_{n+1}) < 2\epsilon$$

whenever a_n and a_{n+1} are adjacent points on Γ (choosing equality for the left inequality whenever possible), and so that each point of Γ on the corner of an S is an a_n. We claim that $\{a_n\}$ is an interpolating sequence. Since we have explicitly made $\rho(a_n, a_m) \geqslant \epsilon$ for $m \neq n$, we need only show that $\mu = \Sigma(\operatorname{Im} a_n)\, \delta_{a_n}$ is a Carleson measure. Since Γ is composed of various edges of our squares S, and since Γ induces a Carleson measure, we need only show that whenever γ is an edge of an S and $(x_0, y_0), \ldots, (x_k, y_k)$ are points on γ for which the adjacent ρ-distance exceeds ϵ, then $\sum_{j=0}^{k} y_j \leqslant \text{const} \mid \gamma \mid$ where the constant may depend on ϵ. Consider first the case when γ is horizontal. Here $x_0 < x_1 < \cdots < x_k$, and $y_0 = y_1 = \cdots = y_k$. We then have

$$\epsilon \leqslant \rho(z_j, z_{j+1}) = \frac{x_{j+1} - x_j}{|(x_{j+1} - x_j) + 2iy_j|} < \frac{x_{j+1} - x_j}{2y_j}.$$

Thus, $y_j < (1/2\epsilon)(x_{j+1} - x_j)$, and so, $\sum_{j=0}^{k} y_j \leqslant (1/2\epsilon) \mid \gamma \mid$. In the case when γ is vertical, we have $x_0 = x_1 = \cdots = x_k$, and $y_0 > y_1 > \cdots > y_k$. Then

$$\epsilon \leqslant \rho(z_j, z_{j+1}) = \frac{y_j - y_{j+1}}{y_j + y_{j+1}} < \frac{y_j - y_{j+1}}{y_j}$$

so that $y_j < (1/\epsilon)(y_j - y_{j+1})$, giving $\sum_{j=0}^{k} y_j \leqslant (1/\epsilon) \mid \gamma \mid$ as above.

Now we have shown that $\{a_n\}$ is an interpolating sequence. Let B^* be the Blaschke product whose zero sequence is $\{a_n\}$. Note that for $z \in \Gamma$, $\mid B^*(z)| < \epsilon$. This follows by observing that $\mid B^*(z)|$ measures the product of the ρ-distances from z to the a_n. At least two of these ρ-distances are bounded by 2ϵ so that the product cannot exceed $4\epsilon^2 < \epsilon$.

We next wish to show that if $z \in Q^0$ and $|f(z)| < 2^{-m}$, then $\mid B^*(z)| < \epsilon$. Points $z \in Q^0$ for which $|f(z)| < 2^{-m}$ will either lie in some S in G or in some T in $\mathscr{G}$. In case $z \in S \in G$, our desired conclusion follows from the maximum principle and the fact that $|B^*| < \epsilon$ on $\partial S \subset \underset{\sim}{Q}^0$. In case $z \in T \in \mathscr{G}$, we no longer must have $\partial T \subset \underset{\sim}{Q}^0$. We do know, though, that $|\partial T \cap \{y = 0\}| = 0$. Consequently $\partial T \cap \{y = 0\}$ has harmonic measure zero relative to $\{y = 0\} \backslash \partial T$ in H^+ and so has harmonic measure zero relative to ∂T on T. Our conclusion now follows from [8, Theorem 1.63].

In summary, we have proved the following lemma.

LEMMA 4. *Let f be any Blaschke product on H^+ and let $\epsilon > 0$. Then there is a corresponding Blaschke product B^* whose zeros are an interpolating sequence in $\underset{\sim}{Q}^0$ such that if $z \in \underset{\sim}{Q}^0$ and $|f(z)| < 2^{-m}$, then $|B^*(z)| < \epsilon$, where m is a universal constant. Also, the zeros of B^* are all in the set $\{z: |f(z)| \leqslant \frac{1}{4}\}$.*

Proof of Theorem 2. Repeated use of Lemma 4 defines the zero set of B^* in $\{z: \frac{1}{2} \leqslant |z| < 1\}$. In $\{z: |z| < \frac{1}{2}\}$ we merely toss in a huge (but finite) number of zeros.

REFERENCES

1. L. V. AHLFORS, "Conformal Invariants: Topics in Geometric Function Theory," McGraw-Hill, New York, 1973.
2. L. CARLESON, An interpolation problem for bounded analytic functions, *Amer. J. Math.* **80** (1958), 921–930.
3. L. CARLESON, Interpolations by bounded analytic functions and the corona problem, *Ann. of Math.* **76** (1962), 547–559.
4. L. CARLESON, The corona theorem, Proceedings of the 15th Scandinavian Congress, Oslo 1968. Lecture Notes in Mathematics, Vol. 118, Springer-Verlag.
5. R. G. DOUGLAS AND W. RUDIN, Approximation by inner functions, *Pacific J. Math.* **31** (1969), 313–320.
6. K. HOFFMAN, "Banach Spaces of Analytic Functions," Prentice-Hall, Englewood Cliffs, New Jersey, 1962.
7. K. HOFFMAN, Bounded analytic functions and Gleason parts, *Ann. of Math.* **86** (1967), 74–111.
8. M. OHTSUKA, "Dirichlet Problem, Extremal Length and Prime Ends" Van Nostrand, New York, 1970.
9. D. WORTMAN, Interpolating sequences on convex curves in the open unit disc, to appear.